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ON THE EXCEPTIONAL CENTRAL SIMPLE NON-LIE MALCEV ALGEBRAS

BY

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ABSTRACT. Malcev algebras belong to the class of binary Lie algebras. Any Lie algebra is a Malcev algebra. In this paper we show that for each seven-dimensional central simple non-Lie Malcev algebra any finite dimensional Malcev module is completely reducible also for positive characteristics. This contrasts with each modular semisimple Lie algebra. As a consequence we get that the classical structure theory for characteristic zero is valid also in the modular case if semisimplicity is replaced by $\mathbf{G}_1$-separability.

The Wedderburn principal theorem is proved for Malcev algebras.

1. Introduction. Structures in algebra and other fields connected with an alternative Cayley algebra show exceptional features. If C is an alternative algebra one recalls that the commutator algebra C^- with the product defined by $a \circ b := a \cdot b - b \cdot a$ is a Malcev algebra. Let D denote a Cayley algebra over a field k with $\operatorname{char}(k) \neq 2, 3$, and e the unit of D. Then any algebra A isomorphic to $D^-/k \cdot e$ is a central simple and non-Lie Malcev algebra and vice versa [4]. A is called an *exceptional Malcev algebra of type* $\mathbf{G}_1$, or of *type* $\mathbf{G}_1$. A is said to be of *type* $\mathbf{C}_M^-$ if A, or equivalently D is split.

Any Lie algebra is a Malcev algebra. Malcev modules are a generalization of Lie modules over Lie algebras. E. J. Taft conjectured that any finite dimensional Malcev module over a Malcev algebra of type $\mathbf{G}_1$ is completely reducible also for positive characteristics. In the following we prove the conjecture for $\operatorname{char}(k) \neq 2, 3$ (Theorem 1). As is well known the analogous statement is false for any simple Lie algebra [3]. If $\operatorname{char}(k) = 0$ the complete reducibility is shown for semisimple Malcev algebras [4]. Our proof applies the classification of irreducible Malcev modules in [1].

The Wedderburn principal theorem was recently extended by E. L. Stitzinger to Malcev algebras if $\operatorname{char}(k) = 0$, and if the radical is $\mathfrak{J}_2$-potent [7]. We prove the theorem for an arbitrary radical R if $\operatorname{char}(k) = 0$, and for the modular case if A/R is $\mathbf{G}_1$-separable.

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In the following we denote by k a field with char$(k) \neq 2, 3$. A and M are presumed finite dimensional k-vector spaces.

2. Definitions. Let A be a binary algebra over k. J: $A \times A \times A \to A$ denotes the *Jacobi map* with $(x, y, z) \mapsto J(x, y, z)$ where $J(x, y, z) := xy \cdot z + yz \cdot x + zx \cdot y \cdot J$ *alternates* if $x^2 := x \cdot x = 0$ for all $x \in A$. Let $x, y, z, t \in A$. We recall that a *Malcev algebra A* is defined by

$$x^2 = 0, \tag{1}$$

and

$$J(x, y, xz) = J(x, y, z)x. \tag{2}$$

Then we have

$$(xy \cdot z)t + (yz \cdot t)x + (zt \cdot x)y + (tx \cdot y)z = ty \cdot xz, \tag{3}$$

$$2tJ(x, y, z) = J(t, x, yz) + J(t, y, zx) + J(t, z, xy) \tag{4}$$

[5]. An A-bimodule is called a *Malcev module* over a Malcev algebra A if the *semidirect sum* or *trivial extension* $E := A \oplus M$ together with the product $(x + m) \cdot (y + m') := xy + xm' + my$ for $m, m' \in M$ is a Malcev algebra. M is called a *Lie module* over A if $xm = -mx$ and $J(x, y, m) = 0$. For a Malcev module M the *module nucleus* N_M defined by $N_M := \{m \in M | \forall x, y \in A: J(x, y, m) = 0\}$ is the maximal Lie submodule. Subsequently, A always denotes a Malcev algebra, and M a Malcev module over A. We define an *A-module homomorphism* f of M into a second A-Malcev module M' by $f(xm) = xf(m)$; if moreover f is injective, f is a *monomorphism* of the A-modules etc.

M is *irreducible* over A if $M \neq \{0\}$, and $\{0\}$ and M are the only submodules over A. If $M = \oplus M_i$, $1 < i < s$, $s \in \mathbf{N}$, each M_i an irreducible submodule, then M is called *completely reducible* over A. Then equivalently for any submodule P there is a submodule N so that $M = P \oplus N$ [3]. If A is canonically considered as an A-bimodule, and M isomorphic to A then M is called *regular*. For A of type $\mathbf{C}_M^-$ the irreducible Malcev modules are up to isomorphism the regular module and the one-dimensional zero module [1].

Let ρ: $A \to \mathrm{End}_k(M)$ denote the canonical representation with $\rho(x)$: $m \mapsto mx$. $k[Y]$ denotes the ring of polynomials in the indeterminate Y. A map φ of A into the subset of irreducible polynomials with $x \mapsto \varphi_x$ is called a *primary function*, a map of A in k a *root*. Then

$$M_\varphi := \left\{m \in M | \forall x \in A \ \exists r \in \mathbf{N}: \left(\varphi_x(\rho(x))\right)^r(m) = 0\right\}.$$

If $\varphi_x = Y - \gamma(x)$ for any x then M_φ is designated by M_γ or $M_\gamma(A)$. Set ${}_1(M_\gamma) := \{m \in M | \forall x \in A: \rho(x)(m) = \gamma(x) \cdot m\}$. M_φ is called a *primary component*, and M_γ a *root space*. If $M_\varphi \neq \{0\}$ then φ is called *characteristic* or *essential* for M, and similarly for roots. A *splits* over M if for any $\rho(x)$ the roots of its minimum polynomial m_x in $k[Y]$ are in k. M is *smooth* if moreover those roots are distinct that is if any m_x is separable over k. A *splitting subalgebra* is defined in the obvious way.

We recall that for any nilpotent splitting subalgebra H there is a *root space decomposition* $A = \bigoplus A_\gamma$ with $\gamma \in \Delta$ [4, Lemma 5]; then

$$A_\beta A_\gamma \subset A_{\beta+\gamma} \quad \text{if } \beta \neq \gamma, \tag{5}$$

$$A_\beta^2 \subset A_{2\beta} + A_{-\beta}, \tag{6}$$

$$J(A_0, A_\beta, A_\beta) \subset A_{-\beta}, \tag{7}$$

$$J(A_0, A_\beta, A_\gamma) = \{0\} \quad \text{if } \beta \neq \gamma, \tag{8}$$

$$J(A_\beta, A_\gamma, A_\delta) = \{0\} \quad \text{if } \beta \neq \gamma \neq \delta \neq \beta, \tag{9}$$

$$J(A_\beta, A_\gamma, A_\gamma) = \{0\} \quad \text{if } \beta \neq 0, \gamma, -\gamma, \tag{10}$$

for $\beta, \gamma, \delta \in \Delta$. A nilpotent subalgebra H of A is called a *Cartan subalgebra* if $H = A_0$ [4]. A is *split* if it has a splitting Cartan subalgebra.

Let $\mathbf{Z}_3$ denote the integers modulo 3, and let the elements of $\mathbf{Z}_3$ be represented by 1, 2, 3. Choose $\nu \in \mathbf{Z}_3$. If A is of type $\mathbf{C}_M^-$ and $\varepsilon \in k \setminus \{0\}$ then A has a basis $T_\varepsilon = \{h, x_\nu, x'_\nu \mid \nu \in \mathbf{Z}_3\}$ with $x_\nu h = \varepsilon x_\nu$, $x'_\nu h = -\varepsilon x'_\nu$, $x_\nu x_{\nu+1} = 2x'_{\nu+2}$, $x'_\nu x'_{\nu+1} = \varepsilon x_{\nu+2}$, $x_\nu x'_\nu = h$, and $x_\nu x'_{\nu+1} = x'_\nu x_{\nu+1} = 0$ [4], [5]. Hence for any ν, $\{h, x_\nu, x'_\nu\}$ is a basis of a split simple three-dimensional Lie (Malcev) algebra B of *type* $\mathbf{A}_1$. Then $\{x_{\nu+1}, x'_{\nu+2}\}$ is the basis of a non-Lie Malcev module of *type* $\mathbf{M}_2$ over B. If H is a splitting Cartan subalgebra of A, and $A_\alpha \oplus H \oplus A_{-\alpha}$ the corresponding root space decomposition, then we may choose T_ε with $H = \langle h \rangle$ and $x_\nu \in A_\alpha$ [4]. The module of type $\mathbf{M}_2$ is up to isomorphism the only non-Lie Malcev module over the Lie algebra of type $\mathbf{A}_1$ [1].

For two algebras B, C over k, $B \oplus. C$ denotes their direct product. Similarly we designate the direct product of two A-submodules M_1 and M_2 by $M_1 \oplus. M_2$. If X is a vector space over k, $x_i \in X$ with $1 < i < r$, $r \in \mathbf{N}$, let $\langle x_1, \ldots, x_r \rangle$ denote the subspace generated by the x_i. For a map $f: X \to Y$, Y a set, let $X^f := f(X)$. For further definitions see [1], [2], [4].

3. The exceptional decomposition of a module. Theorem 1 is preceded by four lemmas.

LEMMA 1. *Let A be a Malcev algebra, H a nilpotent subalgebra, and M a Malcev module over A. If $A = J(A, A, A)$, and $A = \bigoplus A_\pi$ with $\pi \in \Phi$ the primary decomposition over H then for M over H we have*

$$M = \bigoplus M_\pi \quad \text{for } \pi \in \Phi.$$

PROOF. By base field extension we may consider roots instead of characteristic primary functions. Thus let $A = \bigoplus A_\gamma$, $\gamma \in \Delta$, be a H-root space decomposition. Assume $M \neq \bigoplus M_\gamma$, $\gamma \in \Delta$. Then there exists $M_\beta \neq \{0\}$ with $\beta \notin \Delta$. From (8)–(10) then $J(M_\beta, A, A) = \{0\}$. By (4)

$$M_\beta \subset M_\beta A = M_\beta J(A, A, A) \subset J(M_\beta, A, A) = \{0\}.$$

Thus $M_\beta = \{0\}$, proving the lemma. □

Let $h \in A$, $h \neq 0$. If $H = \langle h \rangle$ and α: $H \to k$ a k-linear map we may identify α with $\alpha(h)$. We have

LEMMA 2. *Let A be a Malcev algebra, $h \in A$ with $h \neq 0$, and $H = \langle h \rangle$, M a Malcev module over A. Suppose that A and M are smooth over H. The root spaces are taken over H. Let $A = A_\alpha \oplus H \oplus A_{-\alpha}$ with $\alpha \neq 0$. For $\beta \in \{\alpha, -\alpha\}$ let $M_{2\beta} = \{0\}$. Then for $m \in M_\beta$, $n \in M_0$, and $x, y \in A_\beta$, $y' \in A_{-\beta}$ with $xy' = \delta h$, $\delta \in k$ we get*

$$mx \cdot y' = -2my' \cdot x - 2\beta\delta m, \tag{11}$$

$$mx \cdot y = -m \cdot xy, \tag{12}$$

$$nx \cdot y = -n \cdot xy, \tag{13}$$

$$nx \cdot y' + \beta\delta n \in N_M. \tag{14}$$

PROOF. By (6) $M_\gamma A_\gamma \subset M_{-\gamma}$. For (11) observe

$$\begin{aligned} \beta my' \cdot x &= (mh \cdot y')x \quad \text{and by (3)} \\ &= -(hy' \cdot x)m - (y'x \cdot m)h - (xm \cdot h)y' + xh \cdot my' \\ &= \beta\delta hm + \delta hm \cdot h + \beta xm \cdot y' + \beta x \cdot my' \\ &= -2\beta^2\delta m - \beta mx \cdot y' - \beta my' \cdot x. \end{aligned}$$

Thus $mx \cdot y' = -2my' \cdot x - 2\beta\delta m$.

To obtain (12), consider

$$\begin{aligned} \beta mx \cdot x &= (mh \cdot x)x \quad \text{and again by (3)} \\ &= -(xm \cdot h)x + xh \cdot mx = -2\beta mx \cdot x. \end{aligned}$$

Hence $3\beta mx \cdot x = 0$, therefore, by $\mathrm{char}(k) \neq 3$ and $\beta \neq 0$, $mx \cdot x = 0$. Linearization gives $mx \cdot y = -my \cdot x$. Again applying (3)

$$\begin{aligned} \beta mx \cdot y = (mh \cdot x)y &= -(hx \cdot y)m - (xy \cdot m)h - (ym \cdot h)x + yh \cdot mx \\ &= -\beta m \cdot xy - \beta my \cdot x - \beta mx \cdot y, \end{aligned}$$

and hence $mx \cdot y = -m \cdot xy$, which is (12).

To establish (13), from (3)

$$(nx \cdot x)h = -(xh \cdot n)x + hx \cdot nx = -\beta xn \cdot x - \beta x \cdot nx = 2\beta nx \cdot x.$$

Since $M_{2\beta} = \{0\}$ then $nx \cdot x = 0$. By means of linearization $nx \cdot y = -ny \cdot x$. Through further application of (3)

$$\begin{aligned} nx \cdot y = \beta^{-1} nx \cdot yh &= \beta^{-1}\{(yx \cdot h)n + (xh \cdot n)y + (hn \cdot y)x + (ny \cdot x)h\} \\ &= -n \cdot xy - nx \cdot y - ny \cdot x = -n \cdot xy. \end{aligned}$$

Let $w \in A_{\beta}$, and $w' \in A_{-\beta}$. For (14) we obtain by (3) and (13)

$$\begin{aligned}(nx \cdot y')w &= -(xy' \cdot w)n - (y'w \cdot n)x - (wn \cdot x)y' + wx \cdot ny' \\ &= -\beta\delta nw - (n \cdot wx)y' - ny' \cdot wx = -\beta\delta nw.\end{aligned}$$

Noting (8) then $(nx \cdot y')w' = (ny' \cdot x)w' = -\beta\delta nw'$. Therefore $(nx \cdot y' + \beta\delta n) \cdot A_{\gamma} = \{0\}$ for $\gamma = 0, \alpha, -\alpha$ which proves (14). □

COROLLARY 1. *Let A be split of type* A_1, *M a Malcev module over A, H a splitting Cartan subalgebra of A, $H = \langle h \rangle$, and M smooth for H. For any root β of H with $\beta \neq 0$ and $A_{\beta} \neq \{0\}$ let $M_{2\beta} = \{0\}$.*

Then

$$M = N_M \oplus . J(M, A, A).$$

$J(M, A, A)$ is completely reducible over A.

PROOF. Let $M \neq N_M$. Take a basis $\{x_{\alpha}, x_{-\alpha}, h\}$ for A with $\alpha \in k \setminus \{0\}$, $x_{\alpha}x_{-\alpha} = h$ and $x_{\beta}h = \beta x_{\beta}$ for $\beta \in \{\alpha, -\alpha\}$. By smoothness, M is split over H. Since J alternates and (7)–(9) then $M = N_M + (M_{\alpha} \oplus M_{-\alpha})$. Let $m \in M_{\beta}$ with $mx_{\beta} \neq 0$. By $M_{2\beta} = \{0\}$ from (11) and (12)

$$(mx_{\beta} \cdot x_{-\beta})x_{\beta} = -2\beta\delta mx_{\beta}$$

with $x_{\beta}x_{-\beta} = \delta h \neq 0$. Hence $\langle mx_{\beta} \rangle \oplus \langle mx_{\beta} \cdot x_{-\beta} \rangle$ is an irreducible non-Lie submodule of type $M_2[1]$. Thus $P := M_{\alpha} \cdot x_{\alpha} \oplus M_{-\alpha} \cdot x_{-\alpha}$ is a sum of submodules of type M_2. From (11) we have $M = P + N_M$. Since $J(mx_{\beta}, x_{\beta}, x_{-\beta}) = 3\beta\delta mx_{\beta}$ this sum is direct. Therefore $M = N_M \oplus J(M, A, A)$. The complete reducibility of $J(M, A, A)$ is trivial. □

LEMMA 3. *Let A be of type C_M^-, H a splitting Cartan subalgebra, and M an A-Malcev module.*

Then M is smooth over H.

PROOF. Since $N_M \cdot A = \{0\}$ the assertion is trivial for $M = N_M$. Let $M \neq N_M$, E the semidirect sum of A and M, and $H = \langle h \rangle$. We consider the root spaces over H. By Lemma 1, $M_{\gamma} \neq \{0\}$ implies $A_{\gamma} \neq \{0\}$. Now $E_0 = H \oplus M_0$. Since $J(H, M_0, E) = \{0\}$, from [5, Lemma 5.12] then $HM_0 \subset N_M$.

Thus $HM_0 \cdot A = \{0\}$. By this and (8) it follows that $M_0 A_{\gamma} \subset {}_1(M_{\gamma})$. Observing $A_{\beta} = A_{-\beta}A_{-\beta}$ for $\beta \neq 0$ and (3) one gets ${}_1(M_{\beta})A_{\beta} \subset {}_1(M_{-\beta})$. Hence the sum of H-eigen spaces of M is a submodule.

Let $n \in M_0$, $x \in A_{\beta}$, $x' \in A_{-\beta}$, $\beta \neq 0$, and $xx' = h$. With (14) and observing $nh = nx \cdot x' - nx' \cdot x$ together with $nx' \cdot x \in {}_1(M_0)$ then

$$\beta nx \cdot x' = -(nx \cdot x')x \cdot x' = -(nx' \cdot x)x' \cdot x = \beta nx' \cdot x.$$

Thus $nh = 0$, therefore $M_0 = {}_1(M_0)$. Consider now $m \in M_{\beta}$. We show $m \in {}_1(M_{\beta})$. Assume that $mh \neq \beta m$, and set $\hat{m} := mh - \beta m$. By (8) we then have $\hat{m}x' = mh \cdot x' - \beta mx' = mx' \cdot h = 0$, hence $\hat{m} \cdot A_{-\beta} = \{0\}$. Without restriction let $\hat{m} \in {}_1(M_\beta)$, and h, x_ν, x'_ν with $\nu \in \mathbf{Z}_3$ constitute a basis T_β. For $\mu, \nu \in \mathbf{Z}_3$ with $\mu \neq \nu$ set $x := x_\mu$, $x' := x'_\mu$, $y := x_\nu$. From (11) together with $\hat{m} \cdot A_{-\beta} = \{0\}$, and (3), (12) one derives $2\beta\hat{m}y = -(\hat{m}x \cdot x')y = \beta\hat{m}y$. Thus $\langle \hat{m} \rangle$ is irreducible over A, implying $\beta = 0$ in contradiction to $\beta \neq 0$. Therefore $M_\beta = {}_1(M_\beta)$. □

Lemma 4. *Let A be split of type $\mathbf{G}_1$, H and M as in Lemma* 3, $H = \langle h \rangle$, *and the root spaces taken over H.*

If $\beta \neq 0$, $m \in M_\beta$, and $T_\beta = \{x_\nu, x'_\nu, h | \nu \in \mathbf{Z}_3\}$ then

$$\sum mx'_\nu \cdot x_\nu = -\beta m \quad \textit{for } \nu \in \mathbf{Z}_3. \tag{15}$$

Proof. Set $x := x_1$, $y := x_2$, $z := x_3$, $x' := x'_1$, and similarly for y', z'. We get

$$\begin{aligned} mx' \cdot x &= -\tfrac{1}{2}(yz \cdot m)x \quad \text{and by (3)} \\ &= \tfrac{1}{2}\{(zm \cdot x)y + (mx \cdot y)z + (xy \cdot z)m - xz \cdot ym\} \\ &= my' \cdot y - mz' \cdot z + \beta m + my \cdot y' \quad \text{with (12)}, \\ &= -my' \cdot y - mz' \cdot z - \beta m \quad \text{by (11)}. \end{aligned}$$

Hence $mx' \cdot x + my' \cdot y + mz' \cdot z = -\beta m$ which is (15). □

For a k-vector space and an extension field K of k we define $X_K := X \otimes_k K$. We get

Theorem 1. *Let A be a Malcev algebra of type $\mathbf{G}_1$ and M a Malcev module over A.*

Then M is completely reducible over A.

Proof. For $M = N_M$ the theorem is trivial. Suppose $M \neq N_M$.

(1) Let A be of type $\mathbf{C}_M^-$, and H a splitting Cartan subalgebra, $M = M_\alpha \oplus M_0 \oplus M_{-\alpha}$ the root space decomposition of M over H with $\alpha \neq 0$ according to Lemma 1. If $M = M_0$ then $M = N_M$ by Lemma 3. Thus let $M_\alpha \neq \{0\}$, and $m \in M_\alpha$, $m \neq 0$. From (15) there exists $z' \in A_{-\alpha}$ with $n := mz' \neq 0$. We show that n generates a regular submodule by an argument similar to that in [1]. Take the k-linear map $f: A \to M$ defined by

$$f(h) := n, \qquad f(x) := \beta^{-1} xn \quad \text{if } x \in A_\beta$$

when $\beta \neq 0$. We claim that f is a module homomorphism over A. For $x, y \in A_\beta$, $\beta \neq 0$, we obviously have $xf(h) = f(xh)$, $hf(x) = f(hx)$, and by (13), $xf(y) = f(xy)$. It remains to show for $y' \in A_{-\beta}$ that $y'f(x) = f(y'x)$, equivalently

$$nx \cdot y' = -\beta\delta n \tag{16}$$

where $xy' = \delta h$, $\delta \in k$. We may restrict ourselves to a basis T_β of A corresponding to H with $x_\nu \in A_\beta$. Let $x_\mu x'_\nu = \delta h$, and $x_\mu x'_\lambda = \eta h$ with δ, $\eta \in k$, where $\lambda, \mu, \nu \in \mathbf{Z}_3$. Then by (11) and (12)

$$(mx'_\lambda \cdot x_\mu)x'_\nu = -\tfrac{1}{2}(mx_\mu \cdot x'_\lambda)x'_\nu - \beta\eta mx'_\nu$$
$$= -\tfrac{1}{2}m(x_\mu \cdot x'_\lambda x'_\nu) - \beta\eta mx'_\nu = -\beta\delta mx'_\lambda.$$

We derive the last equality from the multiplication relations for T_β. Hence we have (16). Since $f(h) = n \neq 0$, f is an A-module monomorphism of A in M.

Thus $P := M_\alpha \cdot x'_1 \oplus M_\alpha \oplus M_{-\alpha}$ obviously is a direct sum of regular submodules by (15). For $n' \in M_0$ from (14) then $p := n'x_1 \cdot x'_1 + \beta n' \in N_M$. Hence $M_0 = N_M + P_0$. By $N_M A = \{0\}$ and (16) the sum is direct, and we have $M = N_M \oplus. P$. N_M and P are completely reducible over A, hence M too.

(2) Suppose that A is not split. By [4] A has a Cartan subalgebra H with $H = \langle h \rangle$. Now $A = H \oplus A_\pi$ over H with $\pi_h = Y^2 - c(h)$ and $c(h) \in k \setminus k^2$. Then $M = M_0 \oplus M_\pi$ by Lemma 1. Let K denote a splitting field of π_h.

Set $P := J(M, A, A)$. Then $M = N_M \oplus. P$ by the corresponding decomposition for M_K. For P_0 we choose a basis $\{n_i | 1 < i < s, s \in \mathrm{N}\}$. Let $P(i) := k \cdot n_i \oplus A \cdot n_i$. Then $P(i)_K = K \cdot n_i \oplus A_K \cdot n_i$. $P(i)_K$ is regular over A_K by (1). If f denotes the A_K-module monomorphism of A_K in M_K with $f(h) = n_i$ then $f(A)$ is regular over A. Further

$$f(A) = f(k \cdot h) \oplus f(A \cdot h) = k \cdot f(h) \oplus A \cdot f(h) = P(i).$$

$P(i)$ is irreducible, and $P = \bigoplus. P(i)$ for $1 < i < s$. Hence M is completely reducible over A.

The theorem is proved. □

The following propositions are well known if $\mathrm{char}(k) = 0$ for the semi-simple case [2], [4]. They extend the classical structure theory to positive characteristics for the exceptional case.

PROPOSITION 2. *Let* $A = \bigoplus. A_i$ *with* $1 < i < r$, $r \in \mathrm{N}$, *where each* A_i *is of type* G_1. *Let* M *be a Malcev module over* A.

Then M *is completely reducible over* A.

Moreover $M = N_M \oplus. (\bigoplus. P_j)$ *with* $1 < j < s$, $s \in \mathrm{N}_0$, $N_M A = \{0\}$, *where for any index* j *there is an index* i *so that* P_j *is regular over* A_i, *and* $P_j A_l = \{0\}$ *if* $l \neq i$ *for* $1 < l < r$.

PROOF. If $r = 1$ the first part of the statement is Theorem 1, and the second is a corollary of the proof of Theorem 1. We proceed by induction on the number of simple direct factors of A, and assume that the statement is valid for $r \in \mathrm{N}$. Let now $A = \bigoplus. A_i$ with $1 < i < r + 1$. Set $A' = \bigoplus. A_i$, $2 < i < r + 1$. We choose a Cartan subalgebra H of A. Then $H = H_1 \oplus. H_2$, H_1 a Cartan subalgebra of A_1 and H_2 a Cartan subalgebra of A'.

If $N(1)$ designates the nucleus of M over A_1 then

$$M = N(1) \oplus J(M, A_1, A_1)$$

is a sum of completely reducible A_1-submodules of M. $J(M, A_1, A_1)$ decomposes into a direct sum of regular A_1-submodules. We show that $N(1)$ and $J(M, A_1, A_1)$ are submodules over A. For this we may assume that H is splitting over A. The root spaces of A for H unequal to H are just those of A_1 for H_1 and of A' for H_2 unequal to H_1 and H_2. The corresponding characteristic roots γ are obvious. When $\gamma(H_1) \neq \{0\}$ then $\gamma(H_2) = \{0\}$ thus $A_\gamma \subset A_1$, and vice versa. Applying (4) and Lemma 3 together with (8)–(10) we get

$$J(M, A_1, A_1)A' \subset J(M, A_1, A') = \{0\}.$$

For example if $\beta(H_1) \neq \{0\}$ then $\beta(H_2) = \{0\}$ hence $J(M_\beta, A_\beta, H_2) = \{0\}$ by smoothness of A and M over H_2. If Δ_1 is the set of the characteristic roots of H_1 in A_1, let $(A_1)^1 := \bigoplus (A_1)_\delta(H_1)$ with $\delta \in \Delta_1 \setminus \{0\}$. From (8) for H_1 one has $N(1)A' \cdot (A_1)^1 = \{0\}$. Noting Lemma 3 then $N(1)A' \cdot A_1 = \{0\}$. Thus $N(1)A' \subset N(1)$.

Hence the above yields a direct sum of A-modules. By the induction hypothesis $N(1)$ decomposes as asserted over A'. The proposition is evident. □

The *radical* R of A is by definition the unique maximal solvable ideal. A is called *semisimple* if $R = \{0\}$. *Separability* is defined as usual. In case of $\mathrm{char}(k) = 0$ any semisimple Malcev algebra is separable by the nondegeneracy of the Killing form. A is called $\mathbf{G}_1$-*separable* if there is a base field extension K of k so that the base field extension A_K decomposes into a direct sum of algebras of type $\mathbf{C}_M^-$.

Since the hypothesis of characteristic 0 in the proof of [2, Theorem 2] is only used to establish that M is reducible that proof actually gives the following slightly stronger result

PROPOSITION 3. *Let A be a Malcev algebra and M a Malcev module over A. If A is $\mathbf{G}_1$-separable, then any derivation of A in M is inner.* □

COROLLARY 2. *Let A be a Malcev algebra, and C a $\mathbf{G}_1$-separable subalgebra. Then any derivation of C in A can be extended to an inner derivation of A.* □

For an ideal I of A let $\mathcal{K}^0(I) := I$ and $\mathcal{K}^r(I) := \mathcal{K}^{r-1}(I) \cdot I + (\mathcal{K}^{r-1}(I) \cdot I) \cdot A$ if $r \in \mathbf{N}$. I is called $\mathcal{K}$-*nilpotent* if $\mathcal{K}^r(I) = \{0\}$ for some $r \in \mathbf{N}$ [2]. The *index* $n_{\mathcal{K}}$ of $\mathcal{K}$-nilpotency is the minimal $n_{\mathcal{K}} \in \mathbf{N}$ with $\mathcal{K}^{n_{\mathcal{K}}}(I) = \{0\}$. Nilpotency and $\mathcal{K}$-nilpotency of I are equivalent. The *nilradical* N of A is by definition the maximal nilpotent ideal, hence $N \subset R$. We recall, if B is a subalgebra of A, and $A = B \oplus R$ then this decomposition is called a *Wedderburn* or *Levi decomposition*, and B a *Wedderburn* or *Levi factor* of A.

Similarly as in [2, Theorem 3] we get as a further consequence of Theorem 1

PROPOSITION 4. *Let A be a Malcev algebra with radical R. $n_{\mathcal{K}}$ denotes the index of $\mathcal{K}$-nilpotency of the nilradical of A. Suppose that* $\operatorname{char}(k) > 2n_{\mathcal{K}} - 1$. *Let B be a Levi factor, and C a $\mathbf{G}_1$-separable subalgebra of A.*

Then there is an inner automorphism α of A with

$$C^{\alpha} \subset B. \quad \square$$

COROLLARY 3. *Let A be as in Proposition 4, and A/R $\mathbf{G}_1$-separable. Then any two Levi factors are conjugate by an inner automorphism of A.* $\square$

4. The Wedderburn splitting. Let S be an ideal of A, and $S^2 = \{0\}$. Let φ: $A \to A/S$ with $x \mapsto \underline{x} := x + S$ denote the canonical map. If H is a nilpotent subalgebra of A and γ a linear root of H we define $\underline{\gamma}$: $H^{\varphi} \to k$ by $\underline{\gamma}(\underline{h}) := \gamma(h)$ if $h \in H$. Then obviously

$$(A_{\gamma}(H))^{\varphi} = (A^{\varphi})_{\underline{\gamma}}(H^{\varphi}) \quad \text{and} \quad (A_{\gamma}(H))^{\varphi} = A_{\gamma}(H)/S_{\gamma}(H). \qquad (17)$$

S is a Malcev module over A^{φ} in the canonical way. If $C \subset A^{\varphi}$ denote $C^{\varphi^{-1}} := \varphi^{-1}(C)$. Let $M_{\gamma}(h) := M_{\gamma}(\langle h \rangle)$.

LEMMA 5. *Let S be an ideal of A with $S^2 = \{0\}$, and L an abelian subalgebra of A/S. Furthermore, let S be smooth over L. Then A contains a subalgebra H with $H^{\varphi} = L$ and $H^3 = \{0\}$.*

PROOF. If $\dim(L) = 0$ the assertion is trivial. We use induction on the dimension of L and assume the statement of the lemma for some $n \in \mathbf{N}_0$. Suppose $\dim(L) = n + 1$ and $c \in L$, $c \neq 0$. By the hypothesis of the induction there exists a subalgebra T of A with $T^3 = \{0\}$, $T^{\varphi} \subset L$, $\dim(T^{\varphi}) = n$, and $c \notin T^{\varphi}$. Then $T \subset A_0(T)$, and $L \subset A_0(T)^{\varphi}$ by (17).

We choose $h \in A_0(T)$ with $\underline{h} = c$. Further $S = \bigoplus S_{\gamma}(h)$ for $\gamma \in \Delta$ denotes the root space decomposition over h. Let $h_i \in T$ for $i = 1, \dots, n$, the $\underline{h_i}$ linearly independent. Then $h_i h = \sum {}^i r_{\gamma}$ with $\gamma \in \Delta$ and ${}^i r_{\gamma} \in S_{\gamma}(h) \cap A_0(T)$. Set $h_i^* := h_i - \sum \beta^{-1}\, {}^i r_{\beta}$ for $\beta \in \Delta \setminus \{0\}$. Note $h_i^* \in A_0(T) \cap A_0(h)$. Let H be the subalgebra of A generated by h and the h_i^* for $i = 1, \dots, n$. Hence $H \subset A_0(T) \cap A_0(h)$, and $H^2 \subset S_0(L)$. Thus $H^3 = \{0\}$. $\square$

We prove

THEOREM 5. *Let A be a Malcev algebra over k, R the radical of A, and* $\operatorname{char}(k) = 0$, *or* $\operatorname{char}(k) > 3$. *If* $\operatorname{char}(k) > 3$ *let A/R be $\mathbf{G}_1$-separable.*

Then A decomposes

$$A = B \oplus R$$

where B is a semisimple subalgebra of A with $B \cong A/R$.

PROOF. If $A/R = \{0\}$ or $R = \{0\}$ then the theorem is trivial. Assume that $A/R \neq \{0\}$ and $R \neq \{0\}$. By standard reduction we may assume $R^2 = \{0\}$, and R an irreducible A-Malcev module. Further we may suppose that k is algebraically closed. So $A/R = \oplus . C_i$ with $1 \leq i \leq n$, $n \in \mathbf{N}$, any C_i a simple split subalgebra. In the course of proof we will distinguish different cases. Let $\underline{A} := A/R$, $\underline{x} := x + R$, and $\varphi: x \mapsto \underline{x}$.

(1) Let $\operatorname{char}(k) = 0$, $\underline{A}$ a Lie algebra, and R a Lie module over $\underline{A}$. Then A is a Lie algebra: If $J(A, A, A) = \{0\}$ there is nothing to show. Otherwise $J(A, A, A) = R$. By Lemma 5 there is obviously a Cartan subalgebra H of A so that H^φ is a Cartan subalgebra of $\underline{A}$, and $H^3 = \{0\}$. Decompose A into H-root spaces. Since $H = A_0$, then $J(A_0, A_0, A_0) = \{0\}$. From $\dim((A_\beta)^\varphi) \leq 1$ for $\beta \neq 0$ and R Lie we then have $J(A, A, A) = \{0\}$. Hence A is a Lie algebra for which the theorem is known.

(2) It remains to treat the case that R is not a Lie module over $\underline{A}$, or $\underline{A}$ is not a Lie algebra, or $\operatorname{char}(k) > 3$ with $\underline{A}$ $\mathbf{G}_1$-separable. We proceed by induction on the number n of the simple ideals of $\underline{A}$.

Let $n = 1$. Suppose that $\underline{A}$ is a Lie algebra of type $\mathbf{A}_1$. Let $h \in A$ so that $\langle \underline{h} \rangle$ is a Cartan subalgebra of $\underline{A}$. Decompose A and R over h. Let us consider three cases for R. If R is the one-dimensional zero module then $R = \langle r_0 \rangle$ and $A = A_\alpha \oplus A_0 \oplus A_{-\alpha}$, $\alpha \neq 0$, with $R \subset A_0$. We choose $h' \in A_\alpha A_{-\alpha}$ with $\underline{h}' = \underline{h}$. Then obviously $A_\alpha \oplus \langle h' \rangle \oplus A_{-\alpha}$ is a Levi factor of A.

If R is non-Lie then R is necessarily of type $\mathbf{M}_2$ over $\underline{A}$, and $R = R_\alpha \oplus R_{-\alpha}$ with $R_\beta = \langle r_\beta \rangle$ where $R_\beta \subset A_\beta$ for $\beta \in \{\alpha, -\alpha\}$. Let $x_\beta \in A_\beta$ with $\underline{x}_\beta \neq 0$. Then $J(x_\alpha, x_{-\alpha}, h) = 0$. Any Lie triple of elements generates a Lie subalgebra. Hence $\langle h, x_\alpha, x_{-\alpha} \rangle$ is a Levi factor of A.

Assume third that R is regular over $\underline{A}$. Hence $A = A_\alpha \oplus A_0 \oplus A_{-\alpha}$ with $A_0 = \langle h, r_0 \rangle$, $A_\beta = \langle x_\beta, r_\beta \rangle$ with $\beta \in \{\alpha, -\alpha\}$, and $r_\beta := \beta^{-1} x_\beta r_0$. Note that a canonic $\underline{A}$-module isomorphism is induced by $\underline{h} \mapsto r_0$, and $\underline{x}_\beta \mapsto r_\beta$. Suppose that $\{\underline{x}_\alpha, \underline{x}_{-\alpha}, \underline{h}\}$ is a standard basis for $\underline{A}$. After eventual substitutions $h - \gamma r_0/\alpha$, or $x_{-\alpha} - \delta r_{-\alpha}$ with $\gamma, \delta \in k$, for h or $x_{-\alpha}$ if necessary then $\langle x_\alpha, x_{-\alpha}, h \rangle$ is a Levi factor of A.

Now let $\underline{A}$ be of type $\mathbf{C}_M^-$ and R regular over $\underline{A}$. Take a basis T_α of $\underline{A}$, $T_\alpha = \{y_\nu, y'_\nu, u | \nu \in \mathbf{Z}_3\}$ and set $C := \langle u, y_1, y'_1 \rangle$. $\bar{C}$ is a subalgebra of type $\mathbf{A}_1$. R has a C-decomposition

$$R = B_{1R} \oplus . N_{1R} \oplus . N_{2R},$$

with B_{1R} regular and N_{1R}, N_{2R} of type $\mathbf{M}_2$ over C [1]. In view of the minimal solvable ideals of $C^{\varphi^{-1}}$, and its completely reducible radical, $C^{\varphi^{-1}}$ contains a Levi factor B_1. Let $x, x', h \in B_1$ with $\underline{x} = y_1$, $\underline{x}' = y'_1$, $\underline{h} = u$, and $H := \langle h \rangle$. We decompose A over H into root spaces, $A = A_\alpha \oplus A_0 \oplus A_{-\alpha}$.

We claim $A_\gamma = {}_1(A_\gamma)$. For $\nu \in \{2, 3\}$ choose $x_\nu \in A_\alpha$, $x'_\nu \in A_{-\alpha}$ with $\underline{x}_\nu = y_\nu$, $\underline{x}'_\nu = y'_\nu$. Let $r_0 \in R_0$, $r_0 \neq 0$. If $\beta \neq 0$ and $z \in A_\beta$ set $r_z := \beta^{-1} z r_0$.

Since a Lie triple x_ν, x'_ν, h generates a Lie subalgebra, $x_\nu h = \alpha x_\nu + \delta_\nu r_{x_\nu}$, with $\delta_\nu \in k$. We show $\delta_\nu = 0$. For

$$\begin{aligned}\alpha x_\nu x \cdot x' &= xx_\nu \cdot x'h \\ &= (x'x_\nu \cdot h)x + (x_\nu h \cdot x)x' + (hx \cdot x')x_\nu + (xx' \cdot x_\nu)h \quad \text{by (3)} \\ &= \alpha x_\nu x \cdot x' + \delta_\nu (r_{x_\nu} \cdot x)x' + \alpha x_\nu h - \alpha x_\nu h - \delta_\nu r_{x_\nu} h \\ &= \alpha x_\nu x \cdot x' - 3\alpha\delta_\nu r_{x_\nu},\end{aligned}$$

hence $\delta_\nu = 0$.

Therefore $A_\alpha = {}_1(A_\alpha)$, and equally for $-\alpha$. Thus A is smooth for H.

By Corollary 1, A is completely reducible over B_1. Hence

$$A = B_{1R} \oplus . B_1 \oplus . N_{1R} \oplus . N_{2R} \oplus . N_1 \oplus . N_2$$

with N_1, N_2 of type $\mathbf{M}_2$ over B_1. We may assume $x_2, x_3' \in N_1$ and $x_3, x_2' \in N_2$. If $x_2x_2' = h + \eta r_0$ with $\eta \in k \setminus \{0\}$, replace x_2 by $x_2^* := x_2 - \eta r_{x_2}$. Hence we may suppose $x_2x_2' = h$.

We assert that $B := B_1 \oplus N_1 \oplus N_2$ is an algebra of type $\mathbf{C}_M^-$. We let $y := x_2$, $y' := x_2'$, $z := x_3$, $z' := x_3'$. Then

$$\begin{aligned}yz' &= (2\alpha)^{-1} yh \cdot xy = (2\alpha)^{-1}\{(xh \cdot y)y + (yx \cdot h)y\} \quad \text{by (3)} \\ &= xy \cdot y = -2y \cdot z'.\end{aligned}$$

Thus $yz' = 0$. Similarly $zy' = 0$. Further

$$\begin{aligned}zz' &= (2\alpha)^{-1} x'y' \cdot xy, \quad \text{and with (3)} \\ &= \tfrac{1}{2}\{xx' + yy'\} = h.\end{aligned}$$

From this with (3)

$$yz = \alpha^{-2} z'x' \cdot x'y' = 2x'$$

and similarly $y'z' = \alpha x$. Therefore $B^2 \subset B$. Hence B is a Levi factor of A.

If R is the one-dimensional zero module, take B_1 as before. Similarly A has a B_1-module decomposition

$$A = B_1 \oplus . R \oplus . N_1 \oplus . N_2.$$

By a similar argument one derives that $B := B_1 \oplus N_1 \oplus N_2$ is a Levi factor. Thus the theorem is shown if $\underline{A}$ is of type $\mathbf{A}_1$ or of type $\mathbf{C}_M^-$ when $\operatorname{char}(k) \neq 2$, 3. Let $\operatorname{char}(k) = 0$. Then by [1, Satz 11] we know if $\underline{A}$ is a simple Lie algebra not of type $\mathbf{A}_1$ then R is a Lie module over $\underline{A}$, and the decomposition exists by (1). Hence we have shown the theorem for $\overline{n} = 1$.

We assume as induction hypothesis that the theorem is valid if $\underline{A}$ has exactly n simple direct factors, $n \in \mathbf{N}$. Let $\underline{A} = \oplus . C_i$, $1 < i < n + 1$. By (1) and [1, Satz 11] the remaining part of the proof is obviously reduced to the case that C_1 is either of type $\mathbf{C}_M^-$, or C_1 is of type $\mathbf{A}_1$ with R non-Lie over C_1. In the latter case by the classification R is a module of type $\mathbf{M}_2$ over C_1. Set $G := \oplus . C_i$ with $2 < i < n + 1$. In view of [2, Theorem 1] or of Proposition 2 respectively, we have either $RC_1 = R$ and $RG = \{0\}$, or $RC_1 = \{0\}$.

Let B_1 be a Levi factor of $C_1^{\varphi^{-1}}$, existing by the preceding argument. H_1 denotes a Cartan subalgebra of B_1. Now $RB_1 = R$, or $RB_1 = \{0\}$. In the first case let $\hat{A}_0 := (G^{\varphi^{-1}})_0(H_1)$. Hence $(\hat{A}_0)^\varphi = G$ by (17). By the induction hypothesis $\hat{A}_0$ contains a Levi factor B_2. Take a Cartan subalgebra H_2 of B_2, and set $H := H_1 \oplus . H_2$. Then $H^2 = \{0\}$ by smoothness. We decompose A into H-root spaces $A = \bigoplus A_\gamma$ with $\gamma \in \Delta$, Δ the set of characteristic roots of H in A.

If Δ_i is the set of the characteristic roots of H_i in B_i for $i \in \{1, 2\}$ and $\gamma \in \Delta_i$, let $\gamma^*: H \to k$ be the trivial linear extension with $\gamma^*(h) := \gamma(h)$ if $h \in H_i$, and $\gamma^*(h) = 0$ for $h \in H_j$ if $j \in \{1, 2\}$ and $j \neq i$. Set $\Delta_i^* := \{\gamma^* | \gamma \in \Delta_i\}$. Then $\Delta = \Delta_1^* \cup \Delta_2^*$. Hence $B_2 \subset H_2 \oplus (\bigoplus A_\beta)$, $\beta \in \Delta_2^* \setminus \{0\}$. Observing (17), $B_1 \subset (H_1 \oplus (\bigoplus A_\alpha)) + R$ with $\alpha \in \Delta_1^* \setminus \{0\}$. Because of $RB_2 = \{0\}$ and the composition of the root spaces with (5), $B_1B_2 = \{0\}$. Thus $B := B_1 \oplus . B_2$ is a Levi factor of A.

Finally suppose $RB_1 = \{0\}$. Decompose A as a B_1-module, $A = B_1 \oplus . R \oplus . V$ with $V^\varphi = G$. Since $B_1 V \subset R \cap V$ then $B_1 V = \{0\}$. Further $V \oplus R = G^{\varphi^{-1}}$ is a subalgebra. It contains a Levi factor B_2 by the hypothesis of the induction. Thus $B := B_1 \oplus . B_2$ is a Levi factor of A.

This proves the theorem. □

References

1. R. Carlsson, *Malcev-Moduln*, J. Reine Angew. Math. **281** (1976), 199–210. MR **52** #13968.

2. ———, *The first Whitehead lemma for Malcev algebras*, Proc. Amer. Math. Soc. **58** (1976), 79–84. MR **53** #13337.

3. N. Jacobson, *Lie algebras*, Interscience Tracts in Pure and Appl. Math., no. 10, Interscience, New York, 1962. MR **26** #1345.

4. E. N. Kuzmin, *Malt'sev algebras and their representations*, Algebra i Logika **7** (1968), no. 4, 48–69 = Algebra and Logic **7** (1968), 233–244. MR **40** #5688.

5. A. A. Sagle, *Malcev algebras*, Trans. Amer. Math. Soc. **101** (1961), 426–458. MR **26** #1343.

6. G. B. Seligman, *Modular Lie algebras*, Ergebnisse der Math. und ihrer Grenzgebiete, Band 40, Springer-Verlag, Berlin and New York, 1967. MR **39** #6933.

7. E. L. Stitzinger, *Malcev algebras with J_2-potent radical*, Proc. Amer. Math. Soc. **193** (1975), 1–9. MR **51** #10424.

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